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Hawking radiation as tunneling for extremal and rotating black holes

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Abstract: The issue concerning semi-classical methods recently developed in deriving the conditions for Hawking radiation as tunneling, is revisited and applied also to rotating black hole solutions as well as to the extremal cases. It is noticed how the tunneling method fixes the temperature of extremal black hole to be zero, unlike the euclidean regularity method that allows an arbitrary compactification period. A comparison with other approaches is presented.

Keywords: Space-Time Symmetries, Black Holes.

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1. Introduction

Although several derivations of the Hawking radiation [1] have been proposed in the literature, mostly relying on quantum field theory on a fixed background (see the general references [2]–[5]), it is interesting that only recently a description of black hole radiation and back reaction as a tunneling effect has been investigated semi-classically [6]–[12]. The essence for such calculations is the computation of radial trajectories[1] in the static or stationary region representing the domain of outer communication of the black hole, except that an infinitesimal region behind the event horizon is allowed, which in fact plays a major role. This is because the causal structure of the horizon (it is a null, future directed hyper-surface) makes it impossible to travel from inside to infinity along classically permitted trajectories. This is one way to understand the frequently encountered claim that the horizon radiation must originate from just outside the event horizon itself. In that case it would be quite difficult to detect the imaginary part of the black hole free energy in the effective action formalism, unless one is willing to give up its locality.

In the tunneling approach instead, the particles are allowed to follow classically forbidden trajectories, by starting just behind the horizon onward to infinity. The particles must then travel necessarily back in time, since the horizon is locally to the future of the static or stationary external region. The classical, one-particle action becomes complex, signaling the classical impossibility of the motion, and gives the amplitude an imaginary part which may be considered as a "first quantization" transition amplitude, or equivalently a free field amplitude, knowing that the classical action provides a semi-classical approximation to free field propagators.[2] We stress the fundamentally different physics

[1] The bulk of black holes emission is mainly contained in s-waves.

[2] Investigations of black holes from the point of view of interior degrees of freedom are discussed in [13]

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governing ingoing particles. These can fall behind the horizon along classically permitted trajectories, with a capture cross section of the order of the horizon area. Hence the action for ingoing particles must be real; this will be an important point in our description of black hole radiation.

Given a generic metric representing such a non rotating black hole solution in the static coordinates (Schwarzschild gauge), we have

$$ds^2 = -A(r)dt^2 + B^{-1}(r)dr^2 + C(r)h_{ij}dx^i dx^j \,, \tag{1.1}$$

where the coordinates are labeled as $x^\mu = (t, r, x^i), (i = 1, \ldots, D)$. The metric h_{ij} is a function of the coordinates x^i only, and we shall refer to this metric as the horizon metric. To insure the black hole has finite entropy, we take the horizon to be a compact orientable manifold, say $\mathcal{M}$. Black hole solutions are defined by functions $A(r)$ and $B(r)$ having simple and positive zeroes. This is only a necessary condition to have a black hole; we must also require that the domain of outer communication be "outside of the black hole", i.e. it should correspond to values of the radial coordinate larger than the horizon and extending up to spatial infinity. In most cases $B(r) = A(r)$, and $C(r) = r^2$, but in order to treat more general coordinate system (e.g. isotropic ones) and dilatonic black holes (black holes interacting with scalar fields), we shall also consider $B(r)$ different from $A(r)$ and possibly $C(r)$ not equal to r^2. Interesting black holes with such metrics can be obtained in the Einstein-Maxwell-dilaton coupled system. An example illustrating the first feature is the following two-parameter family solution

$$ds^2 = -\left(1 - \frac{r_+}{r}\right) dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right)^{-1} dr^2 + r^2 dS_2^2 \,, \tag{1.2}$$

where $dS_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ is the round metric on the sphere. The dilaton is $\exp(2(\phi - \phi_\infty)) = (1 - r_-/r)^{-1/2}$; the hole has magnetic charge $q_m = 3r_+r_-/16$, horizon radius $r_+ = 2m$ (which defines m) and can be extended to a non singular, geodesically complete solution with horizons and asymptotically flat infinities. The near horizon geometry in the near extremal limit $r_+ = r_-$ is determined by a two-dimensional black hole of the Jackiw-Teitelboim model, and its entropy can be accounted for exactly by two-dimensional dilaton gravity. As an example illustrating the second feature, we propose the following Kaluza-Klein $4D$ black hole

$$ds^2 = -\Delta^{-1/2}\left(1 - 2mr^{-1}\right) dt^2 + \Delta^{1/2}\left(1 - 2mr^{-1}\right)^{-1} dr^2 + \Delta^{1/2} r^2 dS_2^2 \,, \tag{1.3}$$

where $\Delta = 1 + 2mr^{-1}\sinh^2\gamma$ and γ is a real constant. The metric is asymptotically flat with an event horizon at $r = 2m$. The dilaton is $\exp(-4\phi/\sqrt{3}) = \Delta$ and there is an electric field corresponding to a charge $Q = m\sinh 2\gamma/2$. This metric and dilaton solve the Einstein-Maxwell-dilaton field equations and can be obtained from dimensional reduction of Kaluza-Klein theory, i.e. by finding a solution in five dimensions and then putting the $5D$ metric in the form

$$ds^2 = e^{-4\phi/\sqrt{3}}\left(dx_5 + 2A_a dx^a\right)^2 + e^{2\phi/\sqrt{3}} g_{ab} dx^a dx^b \,.$$

Then g_{ab}, ϕ and A_a are the sought for $4D$ metric, dilaton and gauge field, respectively.

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When the cosmological constant is non positive, often one has only one horizon defined by the largest positive root of the lapse function, and the range of r is an infinite interval. An exception are the topological black holes with higher genus [14]–[16], that have a Cauchy horizon hidden behind the event horizon. When the cosmological constant is positive, there exists the possibility of multiple horizons, some related to black holes, some others to infinity (cosmological horizons) and the range of r is a finite interval.

The form of the metric and choice of coordinates as given above has always been the natural setting to describe Hawking radiation. However, in the tunneling approach of Wilczek and co-workers, the system of coordinates introduced by Painlevé [17] plays a special role. The Painlevé coordinates are associated with a redefinition of the coordinate time, and are given by [17, 18]

$$T = t \pm \int dr \sqrt{\frac{1-B(r)}{A(r)B(r)}} \,. \tag{1.4}$$

As a result, the BH metric becomes a stationary one. In fact a simple computation leads to

$$ds^2 = -A(r)dT^2 \pm 2\sqrt{(1-B(r))\frac{A(r)}{B(r)}}drdT + dr^2 + r^2 h_{ij}dx^i dx^j \,. \tag{1.5}$$

Applying the tunneling method to several BH solutions [10] and [18]–[20] it is found that the tunneling probability at the leading order, for a particle to escape, is given by

$$\Gamma \equiv e^{-\beta_H E} \,, \tag{1.6}$$

where E is the energy of the emitted particle, assumed to be sufficiently small with respect to the total energy of the BH. Of course the flux at infinity will be reduced by the corresponding gray body factors. However, the thermal nature of the flux can be inferred only because the coefficient β_H is known to be the inverse temperature of the hole, so that eq. (1.6) can really be interpreted as a Boltzmann distribution.

Alternatively, the so called method of complex paths has been introduced in several papers [9, 11] (a review can be found in [21]). Here the correct result has been obtained referring to the seminal paper of Hartle and Hawking [22] and making use of several coordinates, including the Schwarzschild ones, but at the expense of the introduction of the concept of multiple mapping, necessary for recovering the covariance. This is because under changes of coordinates within a time slice the results were not invariant.

In this paper, we would like to show that it is possible to work in the Schwarzschild gauge, making use of a variant of the tunneling method, recovering the general covariance simply by observing that a correct use of the theory of distributions in curved (static) space-time requires the introduction of the proper spatial distance, as defined by the spatial metric

$$d\sigma^2 = \frac{d^2 r}{B(r)} + C(r)h_{ij}dx^i dx^j \,, \tag{1.7}$$

a quantity that is *invariant* under redefinitions of the spatial coordinate system, and more generally with respect to the subgroup of the gauge group consisting of time recalibration and spatial diffeomorphisms, defined by [23]–[26]

$$t' = t'\left(t, r, x^i\right) \tag{1.8}$$
$$r' = r'\left(r, x^i\right) \tag{1.9}$$
$$x^{j'} = x^{j'}\left(r, x^i\right). \tag{1.10}$$

As a result, it will not be necessary to work with a complex action for ingoing particles.

The remainder of this paper is organized as follow: in section 2 we show how an ambiguity arises if one insists to treat coordinate singularities as defining distributions, an how it can be resolved by transition to invariant quantities. The role of ingoing particles is also briefly discussed. In sections 3 and 4, we extend the results to a class of extremal black holes and cosmological horizons, respectively. The tunneling method makes it easy to derive the fundamental property of extremal black holes, the vanishing of their temperature. It is worth noticing that the absence of conical singularities in the Wick rotated solution allows for arbitrary compactification of euclidean time. We consider the extension of the method to rotating cases in section 5, especially the three-dimensional BTZ black hole and the Kerr-anti-de Sitter (KAdS) black hole in four dimensions, which also covers the Kerr solution in flat space. Some conclusion and observations are reported in section 6.

2. Hawking radiation as tunneling: single horizon

In this section, in order to illustrate our approach, we will elaborate a variant of the tunneling method in the static case of a single horizon. The multiple horizons case as well as the rotating case will be considered in the next sections.

To begin with, recall the metric we are interested in reads

$$ds^2 = -A(r)dt^2 + B^{-1}(r)dr^2 + C(r)h_{ij}dx^i dx^j . \tag{2.1}$$

We shall consider a scalar particle moving in this classical BH background. Within the semi-classical approximation and being interested only in the leading contribution, we may neglect particle self-gravitation. Thus, the relevant quantity is the classical action I, which satisfies the relativistic Hamilton-Jacobi equation

$$g^{\mu\nu}\partial_\mu I \partial_\nu I + m^2 = 0 , \tag{2.2}$$

which reads

$$-\frac{1}{A(r)}(\partial_t I)^2 + B(r)(\partial_r I)^2 + \frac{1}{C(r)} g^{ij}\partial_i I \partial_j I + m^2 = 0 . \tag{2.3}$$

As usual, due to the symmetries of the metric, one is looking for a solution in the form

$$I = -Et + W(r) + J(x^i) . \tag{2.4}$$

As a consequence

$$\partial_t I = -E , \qquad \partial_r I = W(r) , \qquad \partial_i I = J_i , \tag{2.5}$$

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where J_i are constants (some of them may be chosen to be zero). Thus, the classical action is given by (the sign of the second term corresponds to an outgoing particle, it would be opposite for an ingoing particle)

$$I = -Et + \int \frac{dr}{\sqrt{A(r)B(r)}} \sqrt{E^2 - A(r)\left(m^2 + \frac{g^{ij}J_iJ_j}{C(r)}\right)} + J(x_i)\,. \tag{2.6}$$

It is important to recognize that the action for ingoing particles has to be real, since a particle can fall down in a black hole along a classically permitted trajectory, if only the impact parameter is within the order of magnitude of the horizon radius. Hence the apparent singularity of eq. (2.6) for ingoing particles near the horizon is spurious, and should be eliminated by transition to a coordinate system which is regular on the horizon.[3]

Now one has to face with the key point. If we are dealing with a BH solution, the contribution due to the integral over the radial coordinate is divergent as soon as the integration includes the horizon. One needs a regularization, the natural one is the equivalent of the Feynman prescription and it consists in deforming the contour and, as it is well known, this produces an imaginary part, whose physical consequence is associated with the tunneling process. However, this naive approach leads to an imaginary contribution which is one half the correct one. For example, in the case of the 4-dimensional Schwarzschild BH, $A(r) = B(r) = 1 - \frac{2MG}{r}$, $C(r) = r^2$ and $r_H = 2MG$, a direct computation leads to

$$\operatorname{Im} I = \operatorname{Im} W = \pi r_H E\,. \tag{2.7}$$

Furthermore, if one repeats the above calculation making use of another coordinates system, for example the isotropic ones, defined by (here we consider the case $A(r) = B(r)$, $C(r) = r^2$)

$$t \to t\,, \qquad r \to \rho\,, \qquad \ln\rho = \int \frac{dr}{r\sqrt{A(r)}}\,, \tag{2.8}$$

the metric assumes the form

$$ds^2 = -A(r(\rho))dt^2 + \frac{r^2(\rho)}{\rho^2}\left(d\rho^2 + \rho^2 h_{ij}dx^i dx^j\right). \tag{2.9}$$

In this system of coordinates, the spatial metric is no longer singular at the horizon, and, in the new radial coordinate ρ, has the general form (2.1). For the 4-dimensional Schwarzschild case, the metric in this coordinates is well known and reads

$$ds^2 = -dt^2 \frac{\left(1 - \frac{r_H}{4\rho}\right)^2}{\left(1 + \frac{r_H}{4\rho}\right)^2} + \left(1 + \frac{r_H}{4\rho}\right)^4 (d\rho^2 + \rho^2 dS_2^2)\,. \tag{2.10}$$

This form of the metric is still static, but with a radial part regular at the horizon $\rho = r_H$. We may apply again eq. (2.6) deforming the contour and a direct computation gives the correct result

$$\operatorname{Im} I = \operatorname{Im} W = 2\pi r_H E\,. \tag{2.11}$$

[3]For a somewhat different treatment, see [27].

The reason of this discrepancy can be understood observing that in a curved manifold, the non locally integrable function $1/r$ does not leads to a covariant distribution $1/(r \pm i0)$. One has to make use of the invariant distance defined by eq. (1.7).

If we limit ourselves to the s-wave contribution, only the radial part enters the game and we have for the relevant contribution of the classical action

$$W(\sigma) = \int \frac{d\sigma}{\sqrt{A(r(\sigma))}} \sqrt{E^2 - A(r(\sigma))m^2}\,. \tag{2.12}$$

We may treat the case $A(r)$ different from $B(r)$, but both vanishing at $r = r_H$. Since the leading contribution is coming from the horizon, we may use the following near-horizon approximation,

$$A(r) = A'(r_H)(r - r_H) + \cdots\,, \qquad B(r) = B'(r_H)(r - r_H) + \cdots\,, \tag{2.13}$$

$$\sigma = \int \frac{dr}{\sqrt{B(r)}} = \frac{2}{\sqrt{B'(r_H)}} \sqrt{r - r_H}\,. \tag{2.14}$$

Thus, we have the invariant result

$$W(\sigma) = \frac{2}{\sqrt{A'(r_H)B'(r_H)}} \int \frac{d\sigma}{\sigma} \sqrt{E^2 - A(r(\sigma))m^2}\,. \tag{2.15}$$

The integral is still divergent as soon as $\sigma \to 0$, namely the horizon is reached, but now the prescription corresponding to the Feynman propagator selects the correct imaginary part for the classical action

$$I = \frac{2\pi i}{\sqrt{A'(r_H)B'(r_H)}} E + (\text{real contribution})\,. \tag{2.16}$$

For example, in the 4-dimensional Schwarzschild case, one gets

$$I = 2i\pi r_H E + (\text{real contribution})\,, \tag{2.17}$$

and the semi-classical emission rate, with only the leading term linear in E included, reads

$$\Gamma \equiv e^{-2\,\mathrm{Im}\, I} = e^{-\frac{4\pi E}{\sqrt{A'(r_H)B'(r_H)}}}\,. \tag{2.18}$$

This turns out to coincide with the standard Boltzmann factor as soon as one recognizes that

$$\beta_H = \frac{4\pi}{\sqrt{A'(r_H)B'(r_H)}}\,, \tag{2.19}$$

is the Hawking temperature measured at the infinity for a generic asymptotically flat BH. The same conclusion is also valid for asymptotically anti-de Sitter (AdS) BHs. More attention deserves the asymptotically de Sitter (dS) case, where multiple horizons can be present. We would like to note that β_H can be interpreted as the period of euclidean time in the black hole euclidean section, fixed by the regularity requirement of the absence of a conical singularity at the horizon.

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3. Hawking radiation as tunneling: the extremal case

The physics of extremal black holes has many ramifications extending from classical black hole thermodynamics to string theories, mirrored by the presence of several, radically different solutions. In the following section therefore, we will not cover all the extremal black hole solutions that exist in theories involving coupled gravitational, electromagnetic and scalar fields [28, 29], since for certain values of the dilaton coupling (the term $\exp(-2a\phi)F^2$ in the lagrangian), the thermodynamic description breaks down [30, 31]. To give an illustration of the method regarding extremal black holes (those having vanishing surface gravity), we consider instead some specific examples, starting with the GHS black hole [29] defined by the condition

$$Q^2 = 2M^2 e^{2\phi_0}\,, \tag{3.1}$$

where Q is the charge, M a mass and ϕ_0 is the constant value of the dilaton field. The metric reads

$$ds^2 = -dt^2 + \frac{dr^2}{(1 - C/r)^2} + r^2 dS_2^2\,, \tag{3.2}$$

where $C = 2Me^{-\phi_0}$. The proper distance from a point of radial coordinates r_1 and the horizon $r = C$ is

$$\sigma = r - r_1 - \ln\frac{r - C}{r_1 - C}\,. \tag{3.3}$$

As well known in several extremal cases, the horizon is at infinite spatial distance from a generic point, although it can be traversed in a finite proper time. However, since we have a trivial $A(r) = 1$, the general formula (2.12) leads to

$$W = \sqrt{E - m^2}\int_0^\infty d\sigma\,, \tag{3.4}$$

a divergent integral, whose analytic regularization exists, but no imaginary part is present! The conclusion is that the Hawking temperature vanishes, as was to be expected since without horizon redshift and conserved gauge charges there is no thermal Hawking radiation from the hole. This is in agreement with the results obtained in refs. [32]–[34]. See, however [35].

Furthermore, extremal black holes can still radiate in the charged channel, since they have non zero charge [36]–[38].[4] The process is known as super-radiance and is not associated to any definite temperature. The point is that the tunneling method, as it stands, seems unable to give the super-radiant emission. We also note that for dilaton coupling $a > 0$ the distance of the horizon remains finite; for example this happens in the family

$$ds^2 = -A(r)dt^2 + A(r)^{-1}dr^2 + r^2 B(r)d\omega^2\,,$$

where

$$A(r) = \left(1 - \frac{r_0}{r}\right)^{\frac{2}{1+a^2}}\,, \qquad B(r) = \left(1 - \frac{r_0}{r}\right)^{\frac{2a^2}{1+a^2}}\,.$$

[4] For a different view and conclusions, see [39].

For $a > 1$ even the tortoise coordinate remains finite, so the classical no-hair theorems (relying on the infinite extent of the tortoise coordinate) require other considerations for their validity.

An example with infinite horizon distance is the 5-dimensional topological black hole [14, 40], whose extremal metric reads

$$ds^2 = -dt^2 \frac{(r^2 - r_H^2)^2}{r^2 l^2} + \frac{dr^2}{\frac{(r^2 - r_H^2)^2}{r^2 l^2}} + r^2 dH_3^2 \,, \tag{3.5}$$

where dH_3^3 is the metric associated with a compact hyperbolic manifold. Here we have non trivial $A(r) = B(r)$. The proper distance from a point of radial coordinates r_1 and the horizon $r = r_H$ is

$$\sigma = \frac{l}{2} \left(\ln(r_1^2 - r_H^2) - \ln(r^2 - r_H^2) \right) . \tag{3.6}$$

Again the horizon is located at an infinite distance. Thus,

$$A(\sigma) = \frac{(r_1^2 - r_H^2)^2 e^{-4\frac{\sigma}{l}}}{l^2 r^2(\sigma)} \,. \tag{3.7}$$

Here

$$r^2(\sigma) = r_H^2 + (r_1^2 - r_H^2) e^{-2\frac{\sigma}{l}} \,. \tag{3.8}$$

The formula (2.12) gives,

$$W = \frac{1}{(r_1^2 - r_H^2)} \int_0^\infty d\sigma e^{2\frac{\sigma}{l}} \, r(\sigma) \sqrt{E^2 - m^2 A(r(\sigma))} \,. \tag{3.9}$$

As usual, the integral is divergent as soon as one point arrives at the horizon and no analytic regularization exists giving an imaginary part. As a consequence, once more the Hawking temperature is vanishing. Finally, for solutions with horizon at finite distance one can even obtain infinite temperature, showing that a thermodynamic description can be inadequate.

It should be noted that this correct conclusion can be obtained only if one makes use of the proper distance. In fact, it is easy to show that if one uses the radial coordinate in the discussion of the singular integral, one can find a prescription leading to a non vanishing imaginary part. In fact, in the case of the first example, one would have

$$W = \sqrt{E^2 - m^2} \int_{r_H}^{r_1} dr \frac{1}{(r - r_H)} \,. \tag{3.10}$$

Deforming the contour, one gets a non vanishing imaginary part. One arrives at the same conclusion in the second example we have considered, namely the 5-dimensional extremal topological black hole. Here one would have

$$W = \int_{r_H}^{r_1} dr \frac{r^2}{(r^2 - r_H^2)(r^2 + r_H^2)} \sqrt{E^2 - m^2 A(r)} \,. \tag{3.11}$$

Deforming the contour and making use of the formula

$$\int \frac{dx}{x^2 \pm i0} f(x) = P \int \frac{dx}{x^2} \mp i\pi f'(0) \tag{3.12}$$

one gets again a non vanishing imaginary part.

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4. Hawking radiation as tunneling: multiple horizons

Very recently the so called method of complex paths has been applied to the case of BH solution having multiple horizons [41]. The relevant physical example being $(n+1)$-dimensional Schwarzschild-de Sitter black hole. It is described by the static metric with $-g_{00} = g_{rr}^{-1} = A(r)$, where

$$A(r) = 1 - \frac{\omega_n M}{r^{n-2}} - \frac{r^2}{l^2}\,, \tag{4.1}$$

where ω_n is a geometrical factor containing also the gravitational constant and M can be considered as the mass of the black hole.

This metric is a solution of the vacuum Einstein equations with positive cosmological constant

$$\Lambda = \frac{n(n-1)}{2l^2}\,. \tag{4.2}$$

The case $n = 3$ has been studied in the paper [42].

The geometry of the horizon corresponds to a metric h_{ij} such that

$$R_{ij}(h) = (n-2)\, h_{ij}. \tag{4.3}$$

namely one has spherical black hole solutions in the asymptotically de Sitter space-time, where M is the mass parameter of the black hole. The lapse function $A(r)$ has, at least, two positive simple zeroes, the smaller r_H defines the black hole event horizon, while the larger r_C represents the cosmological event horizon [42]. For example, for $n = 4$, the five dimensional case, one can explicitly find the two roots

$$\begin{aligned} r_H &= \frac{l}{\sqrt{2}}\left(1 - \sqrt{1-4\omega_4 M}\right) \\ r_C &= \frac{l}{\sqrt{2}}\left(1 + \sqrt{1-4\omega_4 M}\right)\,. \end{aligned} \tag{4.4}$$

The background manifold corresponds to $M = 0$ and is the De Sitter space-time, the euclidean counterpart is isometric to S^{n+1}.

Now let us consider a particle located at the classical allowed r, namely with $r_H < r < r_C$. We may repeat the argument for the calculation of the imaginary part of the related classical action. Since we are dealing with two horizons, a particle can be in between by two mutually exclusive tunneling processes associated with the event horizon *or* with the cosmological horizon. These contributions may be evaluated with the method developed in the previous section. Thus, related to the same particle, we have

$$\Gamma_H \equiv e^{-2\,\mathrm{Im}\, I} = e^{-\frac{4\pi E}{A'(r_H)}}\,, \tag{4.5}$$

and

$$\Gamma_C \equiv e^{-2\,\mathrm{Im}\, I} = e^{-\frac{4\pi E}{A'(r_C)}}\,, \tag{4.6}$$

where

$$\beta_H = \frac{4\pi}{A'(r_H)} = \frac{4\pi l^2 r_H}{(n-2)l^2 - n r_H^2}\,, \tag{4.7}$$

and

$$\beta_C = \frac{4\pi}{A'(r_C)} = \frac{4\pi l^2 r_C}{(n-2)l^2 - n r_C^2} \,. \tag{4.8}$$

Since $\beta_H > 0$, it follows that there exists a critical radius

$$r_H < r_0 = l\sqrt{\frac{n-2}{n}} \,. \tag{4.9}$$

at which the two horizons coalesce, and we have the Nariai solution, which represents the largest black hole one can have in de Sitter space. The total tunneling rate should be a mixture of the two contributions with a relative weight proportional to the ratio of the gray body factors of the two horizons. The total flux can probably be described by a flux at some intermediate temperature. The question whether a global unique temperature can be attributed to Schwarzschild-de Sitter space-time has been investigated in [43]–[45], where such a temperature is found at the price of orbifolding the euclidean section. But the quantum state becomes coordinate dependent. It seems that tunneling, by treating separately the horizons could not relate the corresponding temperatures in any significant way.

5. The rotating case

In this section, we shall try to extend the method to some rotating stationary black hole solution.

First, let us consider the simplest case, namely the rotating BTZ black hole [46]

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2\left(d\phi - \frac{J}{2r^2}dt\right)^2 \,, \tag{5.1}$$

written here in units where the three-dimensional Newton constant is $8G = 1$, and l^2 is related to the negative cosmological constant by means of $\Lambda = -l^{-2}$, while the lapse function reads

$$A(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2} \,. \tag{5.2}$$

Here, M and J are respectively the mass and angular momentum of the black hole. When $M^2l^2 > J^2$, one has the event horizon located at

$$r_H^2 = r_0^2\left(1 + \sqrt{1 - \frac{J^2}{M^2l^2}}\right) \,, \tag{5.3}$$

with $r_0^2 = \frac{Ml^2}{2}$. One also has an inner horizon, given by

$$r_I^2 = r_0^2\left(1 - \sqrt{1 - \frac{J^2}{M^2l^2}}\right) \,. \tag{5.4}$$

In the near horizon approximation, one again has

$$A(r) = A'(r_H)(r - r_H) + \cdots \,, \tag{5.5}$$

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r_H^2 d\chi^2 \,, \tag{5.6}$$

in which

$$\chi = \phi - \Omega t, \qquad \Omega = \frac{J}{2r_H^2}\,, \tag{5.7}$$

Ω being the angular velocity of the horizon. As a consequence, we may apply the result of section 2 by writing $I = -Et + J\phi + W(r)$, then passing to the variable χ (which transforms E into $E - \Omega J$) and arrive at

$$\Gamma_{BTZ} \equiv e^{-2\,\mathrm{Im}\,I} = e^{-\frac{4\pi(E-\Omega J)}{A'(r_H)}}\,, \tag{5.8}$$

with $E - \Omega J > 0$ (see below) and

$$A'(r_H) = \frac{l^2 r_H}{2(r_H^2 - r_I^2)}\,. \tag{5.9}$$

The non rotating case corresponds to $J = 0$ and $r_I = 0$, and the extreme case to $r_I = r_H$.

It may be useful to recall that the extreme limit in not unique since, as a rule, there also exists other extremal limits. One can arrive at this by the techniques discussed in several papers. For the rotating case see, for example, [47] and references therein. One alternative extremal limit for the BTZ rotating black hole turns out to be

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r_0^2\left(d\phi - \frac{2r}{r_0 l}\,dt\right)^2\,, \tag{5.10}$$

$$A(r) = \frac{4r^2}{l^2} - \frac{r_0^2}{l^2}\,. \tag{5.11}$$

The horizon is located at $r_H = r_0/2$. We note the absence of an ergosphere, since the natural Killing field ∂_t is time-like everywhere. In fact, the metric has no super-radiant modes and with a linearly diverging local angular velocity it looks like the field of a rotating disk.[5] Again, in the near-horizon approximation, one gets

$$\Gamma_{\mathrm{BTZEX}} \equiv e^{-2\,\mathrm{Im}\,I} = e^{-\frac{4\pi E}{A'(r_H)}}\,, \tag{5.12}$$

with

$$A'(r_H) = \frac{4r_0}{l^2}\,. \tag{5.13}$$

The temperature is $\beta_H = \pi l^2 r_0^{-1}$; thus, in contrast with the former extremal black holes, this one radiates thermally like the more familiar non extremal states.

Along the same lines, one can consider the Kerr-AdS black hole (KAdS) in four dimensions [48]. It is convenient to start from the Kerr-AdS solution written in canonical ADM form, where the lapse and shift functions (the local angular velocity) are seen explicitly

$$ds^2 = -A^2(r,\theta)dt^2 + \frac{dr^2}{B(r,\theta)} + \frac{\rho^2}{\Delta_\theta}d\theta^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2\Xi^2}\,(d\phi - \omega dt)^2\,. \tag{5.14}$$

[5]The study of this metrics may be of interest as a case for AdS/CFT correspondence.

In the following $\Lambda = -3l^{-2}$ is the cosmological constant and

$$A^2(r,\theta) = \frac{\rho^2\Delta_r\Delta_\theta}{\Sigma^2}\,, \qquad B(r,\theta) = \frac{\Delta_r}{\rho^2}\,, \qquad \Xi = 1 - \frac{a^2}{l^2}\,, \tag{5.15}$$

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Delta_r = (r^2+a^2)(1+\frac{r^2}{l^2}) - 2mr\,, \tag{5.16}$$

$$\Sigma^2 = \Delta_\theta(r^2+a^2)^2 - \Delta_r a^2\sin^2\theta\,, \qquad \Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta\,, \tag{5.17}$$

$$\omega = \frac{a\left(\Delta_\theta(r^2+a^2) - \Delta_r\right)\Xi}{\Sigma^2}\,. \tag{5.18}$$

The limiting value of ω as $r \to r_H$ is the angular velocity of the horizon

$$\Omega = \frac{a\Xi}{r_H^2 + a^2}\,. \tag{5.19}$$

The Kerr solution can be obtained in the limit $l \to \infty$, which is the limit of vanishing cosmological constant. The horizon is defined by the largest root of the quartic algebraic equation

$$\Delta_r = 0\,, \qquad (r_H^2 + a^2)\left(1 + \frac{r_H^2}{l^2}\right) - 2mr_H = 0\,. \tag{5.20}$$

Note that the solution is well defined only for $a^2 < l^2$. In the critical limit $a^2 = l^2$, the three dimensional Einstein universe at infinity rotates with the velocity of light. When the horizon angular velocity $\Omega < 1/l$, a global time-like Killing field can be defined outside the event horizon, corresponding to the absence of super-radiance emission. The physical mass of the black hole relative to our choice of time coordinate[6] is $M = m/\Xi$, whereas $J = Ma/\Xi$ is the angular momentum [49, 50].

Let us consider the metric in the near-horizon approximation, since we have seen that it is sufficient to treat the leading effect of Hawking radiation. One has (see, for example [51])

$$\begin{aligned} ds^2 &= -A'(r_H,\theta)(r-r_H)dt^2 + \frac{dr^2}{B'(r_H,\theta)(r-r_H)} + \frac{\rho^2(r_H,\theta)}{\Delta_\theta}d\theta^2 + \\ &\quad + \frac{\Sigma^2(r_H,\theta)\sin^2\theta}{\rho^2(r_H,\theta)\Xi^2}d\chi^2\,, \end{aligned} \tag{5.21}$$

where

$$\chi = \phi - \Omega t\,. \tag{5.22}$$

We also may consider the trajectories with θ and χ constants. Actually, it is a well known property of the Kerr black hole, shared by the KAdS solution, that along geodesics in surfaces $\theta = \theta_0$ constant, the combination $\phi - \Omega t$ is finite on the horizon, while both ϕ and t diverge. As for the BTZ black hole, we may apply again the result of section 2 by writing $I = -Et + J\phi + W(r,\theta)$, then passing to the variable χ (which transforms E into $E - \Omega J$) we arrive at

$$\Gamma \equiv e^{-2\,\mathrm{Im}\,I} = \exp\left(-\frac{4\pi(E-\Omega J)}{\sqrt{A'(r_H,\theta_0)B'(r_H,\theta_0)}}\right)\,. \tag{5.23}$$

[6]One can normalize the Killing vectors so that the corresponding charges generate the $SO(2,3)$ algebra at infinity.

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where $E - \Omega J > 0$ must be assumed. This can be easily understood as follows: the energy and angular momentum of a particle with four-momentum p^a are $E = -p^a K_a$ and $J = p^a \tilde{K}_a$, respectively, where $K = \partial_t$ and $\tilde{K} = \partial_\phi$ is the rotational Killing field. But the Killing field which is time-like everywhere (including the ergosphere) is not K^a, but is instead $\chi = K + \Omega \tilde{K}$. Hence a particle (including those with negative energy inside the ergosphere) can escape to infinity if and only if $p_a \chi^a < 0$, which gives the wanted inequality

$$p^a(K_a + \Omega \tilde{K}_a) = -E + \Omega J < 0$$

At the same time, it is violated only in the super-radiant regime, where the Boltzmann distribution must be replaced with the full Planck distribution, and thus it is outside the reach of the tunneling semi-classical method.

The dependence on the constant angle θ_0 is only apparent, because one finds

$$A'(r_H, \theta_0) B'(r_H, \theta_0) = \frac{r_H^2}{(r_H^2 + a^2)^2} \left(1 + \frac{3r_H^2}{l^2} + \frac{a^2}{l^2} - \frac{a^2}{r_H^2} \right)^2 . \tag{5.24}$$

in agreement with the zeroth law, according to which the surface gravity must be constant all over the horizon. As a result, rearranging a bit, the Hawking temperature is

$$T_H = \frac{3r_H^4 + (l^2 + a^2) r_H^2 - a^2 l^2}{4\pi l r_H (r_H^2 + a^2)} . \tag{5.25}$$

and one can see that $2\pi T_H$ is indeed the surface gravity of the black hole. Here we may note that the tunneling method, having the very nature of a semi-classical scheme, only capture the Boltzmann tail of the Hawking asymptotic flux. In particular, the method misses the super-radiant emission usually associated to rotating non extremal and charged black holes. In this last case this may seem a little bit disappointing, since it is well known [37, 36, 38] that super-radiance is driven by a Schwinger process of pair production, amenable in principle to semi-classical methods. Remarkably, it may not always be the case that Schwinger pair creation takes place through tunneling, as has been shown by Friedmann and Verlinde [52] in their study of pair production of Kaluza-Klein particles in a static KK electric field.

We conclude this section by the following remark. The results obtained within the tunneling method give rise again to Hawking temperature expressions in agreement with the euclidean one, according to which there should be no conical singularity in the associated euclidean, Wick rotated solutions.

6. Conclusions

In this paper, we have revisited the so called tunneling method, namely a simple semi-classical method useful in investigating the Hawking radiation issue. The method has been reformulated in the case of an arbitrary static black hole solution and restricted only to the leading term, namely neglecting the back reaction on the black hole geometry. The different role of ingoing particles has been noticed. Then it has been extended to extremal

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black holes. In these cases, our recipe, consisting in the covariant treatment of the horizon singularity, through the use of spatial proper distance, has allowed to derive the correct result of zero Hawking temperature, but only for those solutions having the horizon at infinite spatial distance. With regard to this issue, we have also stressed that the euclidean method, based on the regularity requirement of absence of conical singularities, works only in the non extremal case, and in this case, the results obtained within the tunneling method give rise again to Hawking temperature expressions in agreement with the euclidean one. The method has also been applied to the multi-horizons case, which is conceptually more difficult to interpret, but the usually accepted Hawking temperatures have been recovered.

We also have considered the stationary (rotating) case. Here we have investigated two important cases, namely the rotating BTZ black hole and the Kerr-AdS black hole solution. Again, making use of the near horizon approximation, the tunneling rate in the leading approximation has been derived and, as a consequence, an expression of the Hawking temperatures agreeing with the values computed by means of standard methods.

Finally, it should be noted that the fundamental thermal nature of the radiation flux remains elusive, and is really inferred from the tunneling method because the coefficient of energy is related to the known temperature of the black hole in precisely the way required by the Boltzmann distribution. A proper treatment should consider also the absorption probabilities, and a check as whether detailed balance between absorption and emission is really satisfied, before inferring a Planck emission spectrum. That this is so, is of course a well known piece of knowledge.

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